

Are fractional Brownian motions predictable?

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Abstract

We provide a device, called the local predictor, which extends the idea of the predictable compensator. It is shown that a fBm with the Hurst index greater than $1/2$ coincides with its local predictor while fBm with the Hurst index smaller than $1/2$ does not admit any local predictor.

1 Introduction

The question in the title is provocative, of course. Everybody familiar with the theory of stochastic processes knows that a continuous adapted process on the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is predictable, in the sense it is measurable with respect to the σ -algebra of *predictable subsets* of $\Omega \times \mathbb{R}^+$. And fractional Brownian motions *are continuous*.

The point is that the predictability has a clear meaning in the discrete time, while in continuous time it loses its intuitive character. Brownian motion serves in many models as a source of unpredictable behavior, but it is predictable in the sense of the general theory of processes.

We are not going to suggest any change in the established terminology, although the old alternative of “well-measurable” sounds more reasonable. Our aim is to provide a device for verifying whether some fractional Brownian Motions are “more predictable” than others.

2 The local predictor and its existence for fBms

We develop the idea of a predictable compensator in somewhat unusual direction. Let, as before, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ be a stochastic basis, satisfying the “usual” conditions, i.e. the filtration $\{\mathcal{F}_t\}$ is right-continuous and \mathcal{F}_0 contains all P -null sets of \mathcal{F}_T . By convention, we set $\mathcal{F}_\infty = \mathcal{F}$.

Let $\{X_t\}_{t \in [0, T]}$ be a stochastic process on (Ω, \mathcal{F}, P) , adapted to $\{\mathcal{F}_t\}_{t \in [0, T]}$ (i.e. for each $t \in [0, T]$, X_t is \mathcal{F}_t measurable) and with càdlàg (or regular) trajectories (i.e. its P -almost all trajectories are right-continuous and possess limits from the left on $(0, T]$).

Suppose we are sampling the process $\{X_t\}$ at points $0 = t_0^\theta < t_1^\theta < t_2^\theta < \dots < t_{k^\theta}^\theta = T$ of a partition θ of the interval $[0, T]$. By the discretization of X on θ we mean the process

$$X^\theta(t) = X_{t_k^\theta} \quad \text{if } t_k^\theta \leq t < t_{k+1}^\theta, \quad X_T^\theta = X_T.$$

If random variables $\{X_t\}_{t \in [0, T]}$ are integrable, we can associate with any discretization X^θ its “predictable compensator”

$$\begin{aligned} A_t^\theta &= 0 \quad \text{if } 0 \leq t < t_1^\theta, \\ A_t^\theta &= \sum_{j=1}^k E(X_{t_j^\theta} - X_{t_{j-1}^\theta} | \mathcal{F}_{t_{j-1}^\theta}) \quad \text{if } t_k^\theta \leq t < t_{k+1}^\theta, \quad k = 1, 2, \dots, k^\theta - 1, \\ A_T^\theta &= \sum_{j=1}^{k^\theta} E(X_{t_j^\theta} - X_{t_{j-1}^\theta} | \mathcal{F}_{t_{j-1}^\theta}). \end{aligned}$$

Notice that A_t^θ is $\mathcal{F}_{t_{k-1}^\theta}$ -measurable for $t_k^\theta \leq t < t_{k+1}^\theta$, and so the processes A^θ are predictable in a very intuitive manner, both in the discrete and in the continuous case. It is also clear, that the discrete-time process $\{M_t^\theta\}_{t \in \theta}$ given by

$$M_t^\theta = X_t^\theta - A_t^\theta, \quad t \in \theta,$$

is a martingale with respect to the discrete filtration $\{\mathcal{F}_t\}_{t \in \theta}$.

If we have square integrability of $\{X_t\}_{t \in [0, T]}$, then the predictable compensator $\{A_t^\theta\}_{t \in \theta}$ possesses also a clear variational interpretation. Fix θ and let \mathcal{A}^θ be the set of discrete-time stochastic processes $\{A_t\}_{t \in \theta}$ which are

$\{\mathcal{F}_t\}_{t \in \theta}$ -predictable, i.e. for each $t = t_k^\theta \in \theta$, $A_{t_k^\theta}$ is $\mathcal{F}_{t_{k-1}^\theta}$ -measurable. Then the predictable compensator $\{A_t^\theta\}_{t \in \theta}$ minimizes the functional

$$\mathcal{A}^\theta \ni A \mapsto E[X - A]_T,$$

where the discrete quadratic variation $[\cdot]$ is defined as usual by

$$[Y]_T = \sum_{t \in \theta} (\Delta Y_t)^2 = \sum_{k=1}^{k^\theta} (Y_{t_k^\theta} - Y_{t_{k-1}^\theta})^2.$$

Now consider a sequence $\Theta = \{\theta_n\}$ of normally condensing partitions of $[0, T]$. This means we assume $\theta_n \subset \theta_{n+1}$ and the mesh

$$|\theta_n| = \max_{1 \leq k \leq k_{\theta_n}} t_k^{\theta_n} - t_{k-1}^{\theta_n} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We will say that an adapted stochastic process $\{X_t\}_{t \in [0, T]}$ with regular trajectories admits a *local predictor* $\{C_t\}_{t \in [0, T]}$ along $\Theta = \{\theta_n\}$ and in the sense of convergence \rightarrow_τ if

$$A^{\theta_n} \rightarrow_\tau C$$

and C has regular trajectories.

As an example we will examine the existence of a local predictor for fractional Brownian motions.

Let us recall that a fractional Brownian motion (fBm) $\{B_t^H\}_{t \in \mathbb{R}^+}$ of Hurst index $H \in (0, 1)$ is a continuous and centered Gaussian process with covariance function

$$E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

For extensive theory of stochastic analysis based on fBms we refer to the most recent monographs [2] and [13].

Theorem 1. *For $H \in (1/2, 1)$ the fractional Brownian motion $\{B_t^H\}_{t \in [0, T]}$ coincides with its local predictor along any sequence of normally condensing partitions and in the sense of the uniform convergence in probability.*

Proof. We consider the natural filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ generated by the fBm $\{B_t^H\}$. Let $\{\theta_n\}$ be a sequence of normally condensing partitions of $[0, T]$

and let $\{A_t^{\theta_n}\}_{t \in \theta_n}$ be the predictable compensator for the discretization of $\{(B^H)_t^{\theta_n}\}$ on θ_n . By the Doob inequality

$$\begin{aligned} E \sup_{t \in \theta_n} ((B^H)_t^{\theta_n} - A_t^{\theta_n})^2 &\leq 4E(B_T^H - A_T^{\theta_n})^2 = 4E[(B^H)^{\theta_n} - A^{\theta_n}]_T \\ &\leq 4E[(B^H)^{\theta_n}]_T = 4 \sum_{k=1}^{k^{\theta_n}} |t_k^{\theta_n} - t_{k-1}^{\theta_n}|^{2H} \\ &\leq 4T|\theta_n|^{2H-1} \rightarrow 0. \end{aligned}$$

Since we have also almost surely

$$\sup_{t \in [0, T]} |(B^H)_t^{\theta_n} - B_t^H| \rightarrow 0,$$

the theorem follows. \square

The above result is a direct consequence of the fact that for $H \in (1/2, 1)$ the fBm is a process of *energy zero in the sense of Fukushima* [6], i.e.

$$E[X^{\theta_n}]_T = E \sum_{k=1}^{k^{\theta_n}} (X_{t_k^{\theta_n}} - X_{t_{k-1}^{\theta_n}})^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Hence we have also

Theorem 2. *If $\{X_t\}$ is continuous adapted and of energy zero in the sense of Fukushima, then it coincides with its local predictor along any sequence of condensing partitions and in the sense of the uniform convergence in probability.*

It may be instructive to write down the assertion of Theorems 1 and 2.

$$\sup_{t \in [0, T]} |X_t - A_t^{\theta_n}| \rightarrow_P 0. \quad (1)$$

Jacod in [9, p. 94], in the context of so-called processes admitting a tangent process with independent increments, introduced a class $B(\{\theta_n\})$ of continuous bounded predictable processes satisfying (1) and

$$\sum_{\{k: t_{k+1}^{\theta_n} \leq t\}} E((X_{t_{k+1}^{\theta_n}} - X_{t_k^{\theta_n}})^2 | \mathcal{F}_{t_k^{\theta_n}}) - (E(X_{t_{k+1}^{\theta_n}} - X_{t_k^{\theta_n}} | \mathcal{F}_{t_k^{\theta_n}}))^2 \rightarrow_P 0. \quad (2)$$

The class $B(\{\theta_n\})_{\text{loc}}$, containing fBms for $H \in (1/2, 1)$, was also considered in [9]. But fBms did not appear in Jacod's paper.

For martingales we have a rather simple statement.

Theorem 3. *The local predictor of a martingale (in particular: of a Brownian motion) trivially exists and equals 0.*

It is interesting that for $H \in (0, 1/2)$ the compensators of discretizations of fBms explode.

Theorem 4. *For $H \in (0, 1/2)$ the fractional Brownian motion $\{B_t^H\}_{t \in [0, T]}$ admits no local predictor. In fact, for any sequence $\{\theta_n\}$ of normal condensing partitions we have*

$$\sup_n E(A_T^{\theta_n})^2 = +\infty.$$

Proof. It suffices to show that

$$\sup_n E(B_T^H - A_T^{\theta_n})^2 = \sup_n E[(B^H)^{\theta_n} - A^{\theta_n}]_T = +\infty. \quad (3)$$

For that we need a lemma, which is basically a result of Nuzman and Poor [14, Theorem 4.4], with corrections due to Anh and Inoue [1, Theorem 1].

Lemma 1. *If $H \in (0, 1/2)$ then for $0 \leq s < t$ there exists a nonnegative function $h_{t,s}(u)$ such that*

$$\int_0^s h_{t,s}(u) du = 1, \quad (4)$$

and

$$E(B_t^H | \mathcal{F}_s) = \int_0^s h_{t,s}(u) B_u^H du, \quad a.s. \quad (5)$$

Recall we work with the natural filtration $\mathcal{F}_s = \sigma\{B_u^H : 0 \leq u \leq s\}$. Note also that it is possible to write down the exact (and complicated) form of the function $h_{t,s}$, but we do not need it.

We need also a remarkably simple lower bound for conditional variances.

Lemma 2. *For $H \in (0, 1/2)$ and $0 \leq s < t$*

$$E(B_t^H - E(B_t^H | \mathcal{F}_s))^2 = E(B_t^H - B_s^H - E(B_t^H - B_s^H | \mathcal{F}_s))^2 \geq \frac{1}{2} |t - s|^{2H}. \quad (6)$$

Proof. Inequality (6) follows from the chain of equalities

$$\begin{aligned}
E(B_t^H - B_s^H - E(B_t^H - B_s^H | \mathcal{F}_s))^2 &= E(B_t^H - B_s^H)^2 - E(E(B_t^H - B_s^H | \mathcal{F}_s))^2 \\
&= E(B_t^H - B_s^H)^2 - E((B_t^H - B_s^H)E(B_t^H - B_s^H | \mathcal{F}_s)) \\
&= E(B_t^H - B_s^H)^2 - E(B_t^H E(B_t^H | \mathcal{F}_s)) - E(B_s^H)^2 \\
&\quad + E(B_s^H E(B_t^H | \mathcal{F}_s)) + E B_t^H B_s^H \\
&= (t - s)^{2H} - \frac{1}{2} \int_0^s h_{t,s}(u)(t^{2H} + u^{2H} - (t - u)^{2H}) du - s^{2H} \\
&\quad + \frac{1}{2} \int_0^s h_{t,s}(u)(s^{2H} + u^{2H} - (s - u)^{2H}) du + \frac{1}{2}(t^{2H} + s^{2H} - (t - s)^{2H}) \\
&= \frac{1}{2}(t - s)^{2H} + \frac{1}{2} \int_0^s h_{t,s}(u)((t - u)^{2H} - (s - u)^{2H}) du,
\end{aligned}$$

and from the observation that for $H \in (0, 1/2)$

$$\frac{1}{2} \int_0^s h_{t,s}(u)((t - u)^{2H} - (s - u)^{2H}) du \geq 0.$$

□

Now we are ready to verify (3). By (6)

$$E[(B^H)^{\theta_n} - A^{\theta_n}]_T \geq \frac{1}{2} \sum_{k=1}^{k^{\theta_n}} |t_k^{\theta_n} - t_{k-1}^{\theta_n}|^{2H} \rightarrow +\infty,$$

for every sequence $\{\theta_n\}$ of normal condensing partitions of $[0, T]$. □

Remark 2.7 The random variables $A_T^{\theta_n}$ are Gaussian, so $\sup_n E(A_T^{\theta_n})^2 = +\infty$ is equivalent to the lack of tightness of the family $\{A_T^{\theta_n}\}$. Thus in the case $H \in (0, 1/2)$ the compensators do not stabilize in any reasonable probabilistic sense.

3 On the existence of local predictors

3.1 Submartingales

It is not difficult to show that any continuous and nondecreasing adapted integrable process coincides with its local predictor in the sense of the uniform

convergence in probability. This implies in turn that any submartingale of class D with continuous increasing process in the Doob-Meyer decomposition also admits a local predictor which coincides with its predictable continuous compensator.

This is no longer true if the compensator is discontinuous. We have then in general only weak in L^1 convergence of discrete compensators. Such convergence, although satisfactory from the analytical point of view, brings only little probabilistic understanding to the nature of the compensation.

To overcome this difficulty, the author proposed in [10] an approach based on the celebrated Komlós theorem [12]. It is proved *ibidem* that given any sequence $\{\theta_n\}$ of partitions one can find a subsequence $\{n_j\}$ along which the Césaro means of compensators of discretizations converge to the limiting compensator. More precisely, if $\{n_j\}$ is the selected subsequence and we denote by $\{A_t^j\}$ the predictable compensator of the discretization on θ_{n_j} , then for each rational $t \in [0, T]$

$$B_t^N = \frac{1}{N} \sum_{j=1}^N A_t^j \rightarrow A_t, \quad \text{a.s.}, \quad (7)$$

where A is the continuous-time process in the Doob-Meyer decomposition. In fact the above convergence can be strengthened: for each stopping time $\tau \leq T$ we have

$$\limsup_{N \rightarrow +\infty} B_\tau^N = A_\tau, \quad \text{a.s.} \quad (8)$$

In particular, this directly implies predictability of $\{A_t\}$.

3.2 Processes with finite energy and weak Dirichlet processes

Graversen and Rao [8] proved the Doob-Meyer type decomposition for a wide class of *processes with finite energy*. Examples of how such decomposition can work in the framework of *weak Dirichlet processes* (including cases of uniqueness) were provided in several recent papers (see [3], [4], [5], [7]). Similarly as in the general theory for submartingales, in the Graversen-Rao original paper the existence of the predictable decomposition was obtained by the weak- L^2 arguments.

The author proved in [11] that the Komlós machinery works perfectly also in this problem. For a sequence $\{\theta_n\}$ of partitions of $[0, T]$ such that random

variables $\{A_T^{\theta_n}\}$ are *uniformly integrable* one can select a subsequence such that for each stopping time $\tau \leq T$

$$B_\tau^N \rightarrow A_\tau, \quad \text{in } L^1.$$

In the above we use the setting of (7) and (8).

In [11] an example of a bounded process was given, for which the terminal values $\{A_T^{\theta_n}\}$ were not uniformly integrable. It follows from our Theorem 4 that the fractional Brownian motion with the Hurst index $H \in (0, 1/2)$ is another, more natural example of such phenomenon.

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